Combinatorial properties of degree sequences of 3-uniform hypergraphs arising from saind sequences

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Outline for section

1 Main notions and State of the art

2 Our findings: degree sequences of 3-uniform hypergraphs arising from saind sequences

3 Conclusions and future developments

Hypergraphs

Definition (Hypergraph)

A **hypergraph** $\mathcal{H}$ is defined as a couple (V, E), where V is a finite set of vertices $v_1, \ldots, v_n$, and $E \subset 2^{|V|} \backslash \emptyset$ is a set of hyperedges, i.e. a collection of subsets of V.

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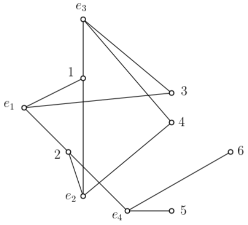
A hypergraph is **simple** if it has no loop and no equal hyperedges.



The notion of degree sequence

Definition (Degree of a vertex)

Given an hypergraph $\mathcal{H} = (V, E)$, the **degree of a vertex** $v \in V$ is the number of hyperedges that contain v.

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Definition (Degree sequence)

Given a hypergraph $\mathcal{H} = (V, E)$, the **degree sequence** of $\mathcal{H}$ is $(d_1, d_2, \ldots, d_n)$, where $d_1 \geq d_2 \geq \cdots \geq d_n$ are the degrees of the vertices.

Starting problem

Starting Problem: k-Seq

Given $\pi = (d_1, d_2, \ldots, d_n)$ a non decreasing sequence of positive integers, can π be the degree sequence of a $k-$uniform simple hypergraph?

State of the art

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- 2018 → **Deza** et al. proved that for any fixed integer $k \geq 3$ the problem is $NP-$complete, so assuming $P \neq NP$ it seems also intractable to find a good characterization for π even for the simplest case of $3-$uniform hypegraphs.

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New Goal

Assuming $P \neq NP$, find which instances are really $NP-$complete and which, instead, are solvable in polynomial time.

The matrix M_S

- Let $S = (s_1, \ldots, s_k)$ be an array of integers.
- We define a binary matrix M_S of dimension $k' \times k$ collecting all the distinct rows (arranged in lexicographical order) that satisfy the following constraint: for every index i, the i-th row of M_S has all elements equal to zero except three entries in positions j_1, j_2 and j_3 such that $s_{j_1} + s_{j_2} + s_{j_3} > 0$.

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For instance, the matrix M_S of $S = (5, 2, 2, -1, -4, -4)$ is

5	2	2	−1	−4	−4
1	1	1	0	0	0
1	1	0	1	0	0
1	1	0	0	1	0
1	1	0	0	0	1
1	0	1	1	0	0
1	0	1	0	1	0
1	0	1	0	0	1
0	1	1	1	0	0
7	5	5	3	2	2

The matrix M_S

- M_S can be regarded as the **incidence matrix** of a (simple) 3-uniform hypergraph $\mathcal{H}_S = (V, E)$ such that the element $M_S(i, j) = 1$ if and only if the hyperedge $e_i \in E$ contains the vertex v_j.

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- Let $\pi_S = (p_1, \dots, p_k)$ denote the degree sequence of $\mathcal{H}_S$. It holds $\sum_{i=1}^{k'} M_S(i, j) = p_j$.

Problems

Problem 1

Determine the computational complexity of 3-Seq restricted to the class of the instances π_S.

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Problem 2

Characterize the 3-sequences whose related 3-uniform hypergraphs are unique up to isomorphism. Determine the computational complexity of 3-Seq restricted to that class of instances.

Saind arrays

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For any $n \geq 2$, the **saind array** of size n is an integer array $S(n) = (n, n-1, n-2, \ldots, 2-2n)$.

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Index	1	2	3	4	5	6	7	8
$s_3=$	3	2	1	0	-1	-2	-3	-4
	1	1	1	0	0	0	0	0
	1	1	0	1	0	0	0	0
	1	1	0	0	1	0	0	0
	1	1	0	0	0	1	0	0
	1	1	0	0	0	0	1	0
	1	1	0	0	0	0	0	1
	1	0	1	1	0	0	0	0
	1	0	1	0	1	0	0	0
	1	0	1	0	0	1	0	0
	1	0	1	0	0	0	1	0
	1	0	0	1	1	0	0	0
	1	0	0	1	0	1	0	0
	0	1	1	1	0	0	0	0
	0	1	1	0	1	0	0	0
	0	1	1	0	0	1	0	0
	0	1	0	1	1	0	0	0
$v_3=$	12	10	8	6	5	4	2	1

Queue and Saind sequence

Queue of v_n

- $n = 2 \rightarrow \pi(2) = (4, 3, 2, \mathbf{2}, \mathbf{1})$

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- $n = 4 \rightarrow \pi(4) = (25, 21, 18, 15, 12, 10, \mathbf{9}, \mathbf{6}, \mathbf{4}, \mathbf{2}, \mathbf{1})$

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- $n = 5 \to$
  $\pi(5) = (42, 37, 32, 28, 24, 20, 17, 15, \mathbf{12},\ \mathbf{9},\ \mathbf{6},\ \mathbf{4},\ \mathbf{2},\ \mathbf{1})$

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- As n increases, the entries of $Q(n)$ give rise to an infinite sequence: the **Saind sequence** $(w_n)_{n \geq 1}$.
- The first few terms of w_n are:
  $1, 2, 4, 6, 9, 12, 16, 20, 25, 30, 36, 42, 49, 56, 64, 72, 81, 90, 100 \ldots$

Queue triads

Index	1	2	3	4	5	6	7	8
$s_3=$	3	2	1	0	-1	-2	-3	-4
	1	1	1	0	0	0	0	0
	1	1	0	1	0	0	0	0
	1	1	0	0	1	0	0	0
	1	1	0	0	0	1	0	0
	1	1	0	0	0	0	1	0
	1	1	0	0	0	0	0	1
	1	0	1	1	0	0	0	0
	1	0	1	0	1	0	0	0
	1	0	1	0	0	1	0	0
	1	0	1	0	0	0	1	0
	1	0	0	1	1	0	0	0
	1	0	0	1	0	1	0	0
	0	1	1	1	0	0	0	0
	0	1	1	0	1	0	0	0
	0	1	1	0	0	1	0	0
	0	1	0	1	1	0	0	0
$v_3=$	12	10	8	6	5	4	2	1

The queue-triads are: $(1, 2, 6), (1, 3, 6), (1, 4, 6), (2, 3, 6)$.

Queue triads

Queue triads of size n and pointer k can be computed by the following algorithm:

Algorithm 1 Algorithm that calculates queue-triads

Input: n

Output: All the queue-triads of size n

Step 1: We determine the pointers: $\begin{cases} k_o = 3 \cdot \frac{n+1}{2} & \text{if } n \text{ is odd} \\ k_e = \frac{3n+2}{2} + 1,\ k_e' = \frac{3n+2}{2} & \text{if if } n \text{ is even} \end{cases}$

Step 2: We calculate the values of i for the pointers determined in Step 1:

- n odd: $\begin{cases} 1 \leq i \leq \frac{3 \cdot n - k}{2} & k_o \text{ odd} \\ 1 \leq i \leq \frac{3 \cdot n - k + 1}{2} & k_o \text{ even} \end{cases}$

- n even, and $k \in \{k_e, k_e'\}$: $\begin{cases} 1 \leq i \leq \frac{3 \cdot n - k + 1}{2} & k \text{ odd} \\ 1 \leq i \leq \frac{3 \cdot n - k}{2} & k \text{ even} \end{cases}$

Step 3: We calculate j: $i + 1 \leq j \leq 3 \cdot n - k - (i - 2)$.

OEIS and A002620

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

a002620 Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **a002620**

Displaying 1-10 of 366 results found. page 1 2 3 4 5 6 7 8 9 10 ... 37

Sort: relevance | references | number | modified | created Format: long | short | data

A002620 Quarter-squares: floor(n/2)*ceiling(n/2). Equivalently, floor(n^2/4). +40 366

(Formerly M0998 N0374)

0, 0, 1, 2, 4, 6, 9, 12, 16, 20, 25, 30, 36, 42, 49, 56, 64, 72, 81, 90, 100, 110, 121, 132, 144, 156, 169, 182, 196, 210, 225, 240, 256, 272, 289, 306, 324, 342, 361, 380, 400, 420, 441, 462, 484, 506, 529, 552, 576, 600, 625, 650, 676, 702, 729, 756, 784, 812 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,4

COMMENTS

b(n) = **A002620**(n+2) = number of multigraphs with loops on 2 nodes with n edges [so g.f. for b(n) is 1/((1-x)^2*(1-x^2))]. Also number of 2-covers of an n-set; also number of 2 X n binary matrices with no zero columns up to row and column permutation. - Vladeta Jovovic, Jun 08 2000

a(n) is also the maximal number of edges that a triangle-free graph of n vertices can have. For n = 2m, the maximum is achieved by the bipartite graph K(m, m); for n = 2m + 1, the maximum is achieved by the bipartite graph K(m, m + 1). - Avi Peretz (njk(AT)netvision.net.il), Mar 18 2001

a(n) is the number of arithmetic progressions of 3 terms and any mean which can be extracted from the set of the first n natural numbers (starting from 1). - Santi Spadaro, Jul 13 2001

This is also the order dimension of the (strong) Bruhat order on the Coxeter group A_{n-1} (the symmetric group S_n). - Nathan Reading (reading(AT)math.umn.edu), Mar 07 2002

Let M_n denote the n X n matrix m(i,j) = 2 if i = j; m(i, j) = 1 if (i+j) is even; m(i, j) = 0 if i + j is odd, then a(n+2) = det M_n. - Benoit Cloitre, Jun 19 2002

Sums of pairs of neighboring terms are triangular numbers in increasing order. - Amarnath Murthy, Aug 19 2002

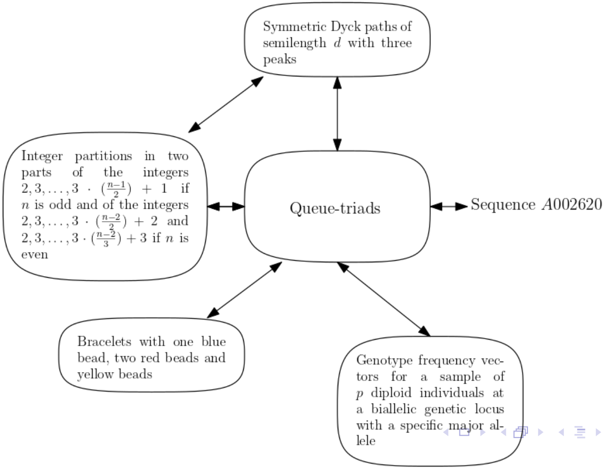
Also, from the starting position in standard chess, minimum number of captures by pawns of the same color to place n of them on the same file (column). Beyond a(6), the board and number of pieces available for capture are assumed to be extended enough to accomplish this task. - Rick L. Shepherd, Sep 17 2002

For example, a(2) = 1 and one capture can produce "doubled pawns", a(3) = 2 and two captures is sufficient to produce tripled pawns, etc. (Of course other, uncounted, non-capturing pawn moves are also necessary from the starting position in order to put three or more pawns on a given file.) -

Saind sequence and A002620

Theorem

For any $m \geq 1$, we have $w_m = \left\lfloor \frac{m+1}{2} \right\rfloor \cdot \left\lceil \frac{m+1}{2} \right\rceil$.



Integer partitions

Definition (Integer partitions)

A **partition of a positive integer** n is a sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_m)$, such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_m = n$.

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A summand in a partition is called a **part**.
If $P(i,k)$ is the number of integer partitions of i into k parts, and if $k = 2$, then

$$a(n) = \sum_{i=2}^{n} P(i,2)$$

where $a(n)$ is the $n-th$ number of the sequence $A002620$.

Integer partitions and Queue triads

Proposition

For any $n \geq 2$, there is a bijection between the family of queue triads of size n and pointer k and the one of integer partitions in two parts of the integers $2, 3, \ldots, k-2$.

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We define the function f as follows: given a queue triad $t = (x, y, k)$, the corresponding integer partition $f(t) = (g, p)$ is obtained by setting $g = y - 1$ and $p = x$.

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Example: From queue-triads to integer partitions in 2 parts

$(1, 2, 6) \rightarrow (1, 1)$
Indeed:

$$g = 2 - 1 = 1$$

$$p = 1$$

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Example: From queue-triads to integer partitions in 2 parts

$(1, 4, 6) \rightarrow (3, 1)$
Indeed:

$$g = 4 - 1 = 3$$

$$p = 1$$

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Example: From queue-triads to integer partitions in 2 parts

$(2, 3, 6) \rightarrow (2, 2)$
Indeed:

$$g = 3 - 1 = 2$$

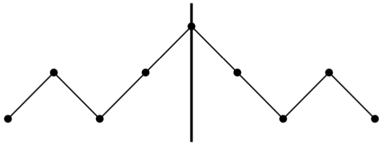
$$p = 2$$

Symmetric Dyck paths with 3 peaks

Definition (Dyck path)

A **Dyck path** of semi-length n is a path P of length $2n$ in the positive quarter plane that uses *UP* steps $U = (1, 1)$ and *DOWN* steps $D = (1, -1)$ starting at the origin and returning to the x-axis.

A particular subclass of Dyck paths is formed by Dyck paths of length $2n$ that are **symmetric**, which means that they are symmetrical with the respect to the axis of symmetry which passes through the upper end of the $n-th$ step and it is parallel to the y-axis.



Symmetric Dyck paths with 3 peaks and Queue triads

Proposition

For any n, the family of queue triads with size n and pointer k is in bijection with symmetric Dyck paths with exactly three peaks and semi-length $\ell = (3n - 1) - k + 3$.

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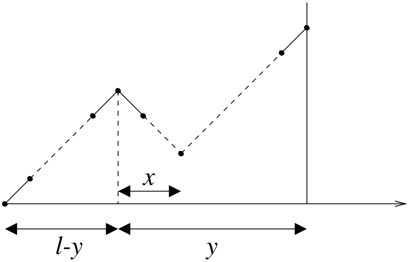

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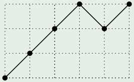
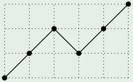
Example: From queue-triads to prefixes

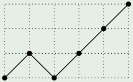
For example, the queue triads of size 3 and pointer 6, i.e. $(1,2,6)$, $(1,3,6)$, $(1,4,6)$, $(2,3,6)$, are mapped onto the paths:



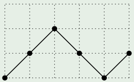
(1,2,6)



(1,3,6)

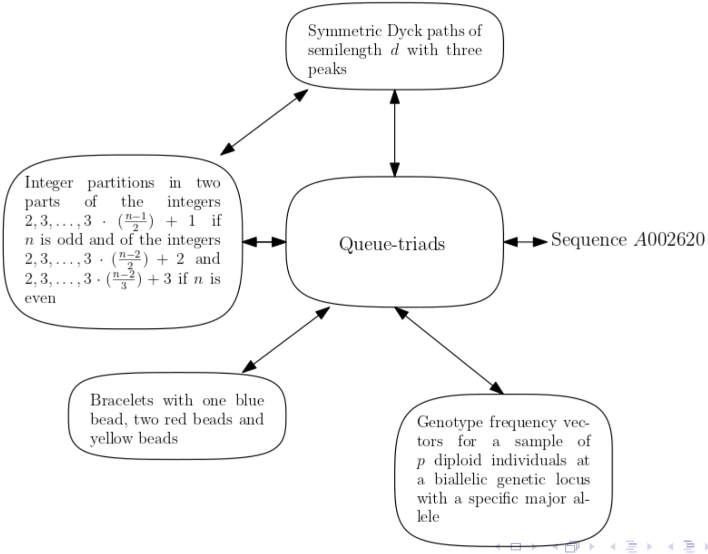


(1,4,6)



(2,3,6)

Bijections in the case of 3−uniform hypergraphs



Future developments

Study of sequences similar to Saind arrays, starting with a slightly different array and analyzing the combinatorial properties of the corresponding degree sequences.

Array	Number sequence	First terms
$(n, n, n-1, n-1, \dots, 1-2n, 1-2n)$	A035608	$1, 5, 10, 18, 27, 39, 52, 68, 85, \dots$
$(n, n, n, \dots, -n, -n, -n)$	A079079	$3, 6, 12, 24, 42, 63, 90, 120 \dots$

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