Compositional Linear Programming

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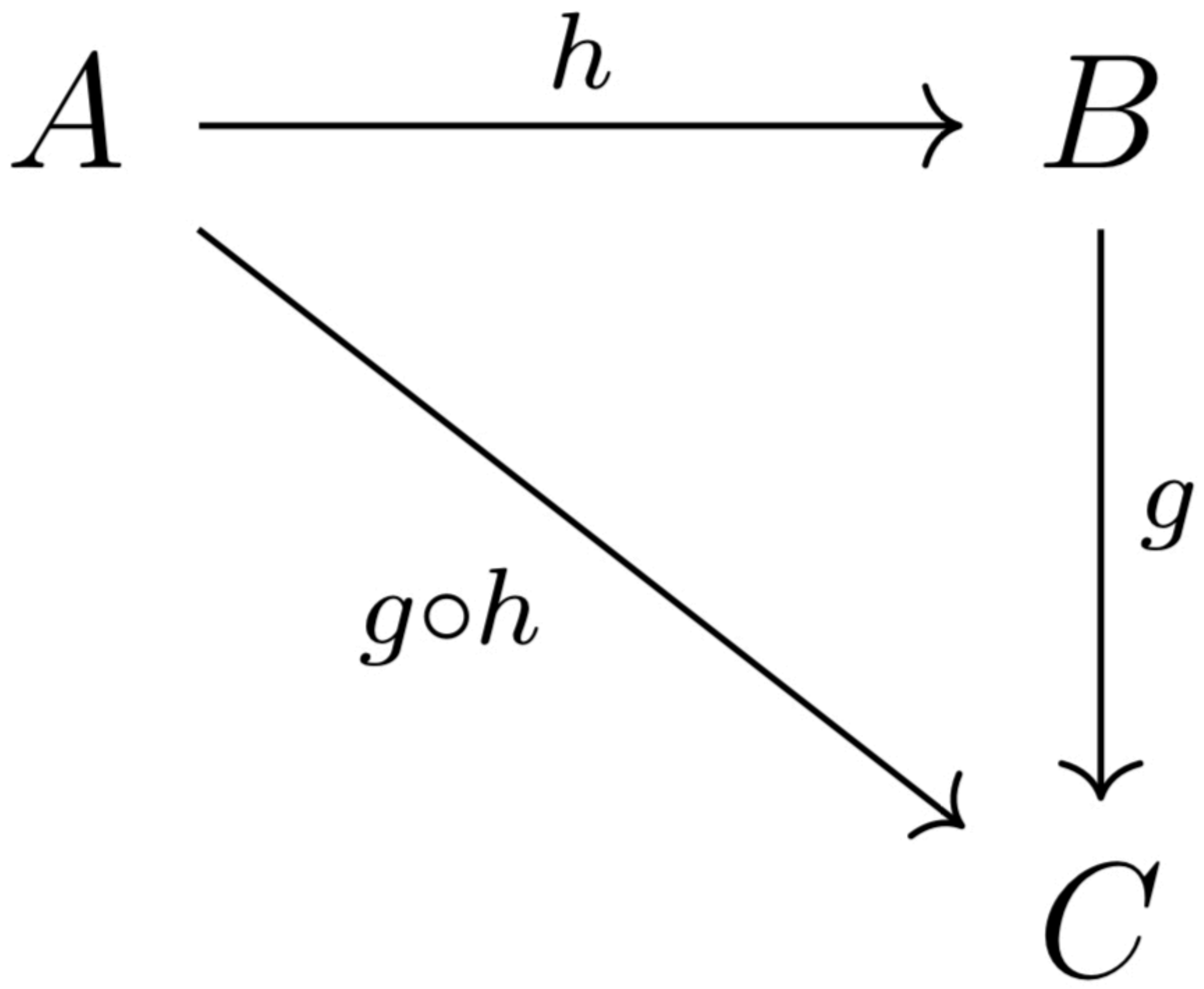
Compositionality

The nature of **complex structures** is entirely determined by that of their **simpler parts** and the way these are **composed**

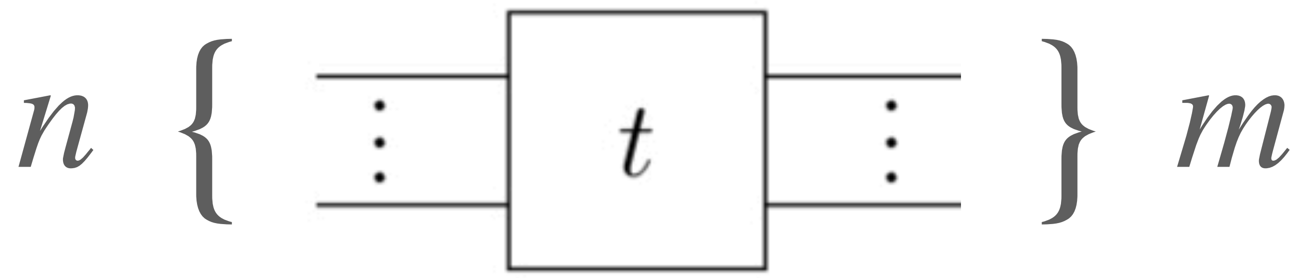
Category Theory



Category Theory

- **Set** is the category with *sets* as objects and *functions* as arrows.
- **Grp** is the category with *groups* as objects and *homomorphisms* as arrows.
- **Top** is the category with *topological spaces* as objects and *continuous functions* as arrows.
- **Ab** is the category with *abelian groups* as objects and *group homomorphisms*, which preserve the abelian structure, as arrows.

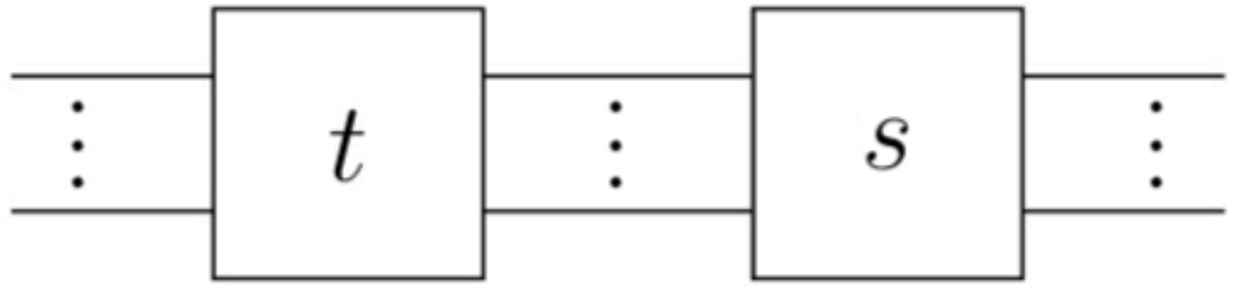
PROPs



$$n \longrightarrow m$$

Sequential composition

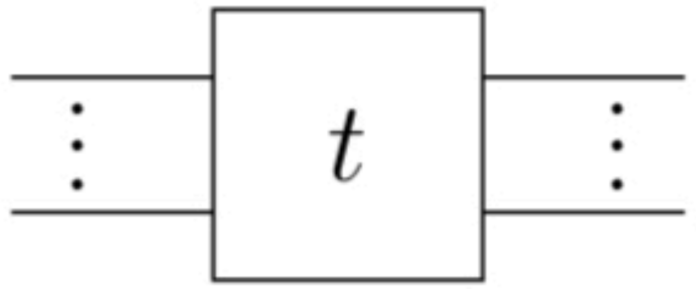
$n \xrightarrow{t} m \qquad m \xrightarrow{s} o$



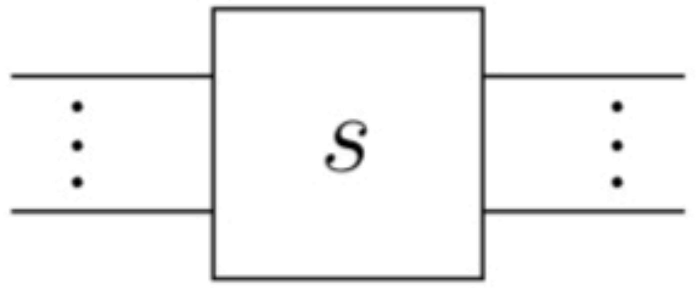
$$n \xrightarrow{t;\, s} o$$

Parallel composition

$n \xrightarrow{t} m$

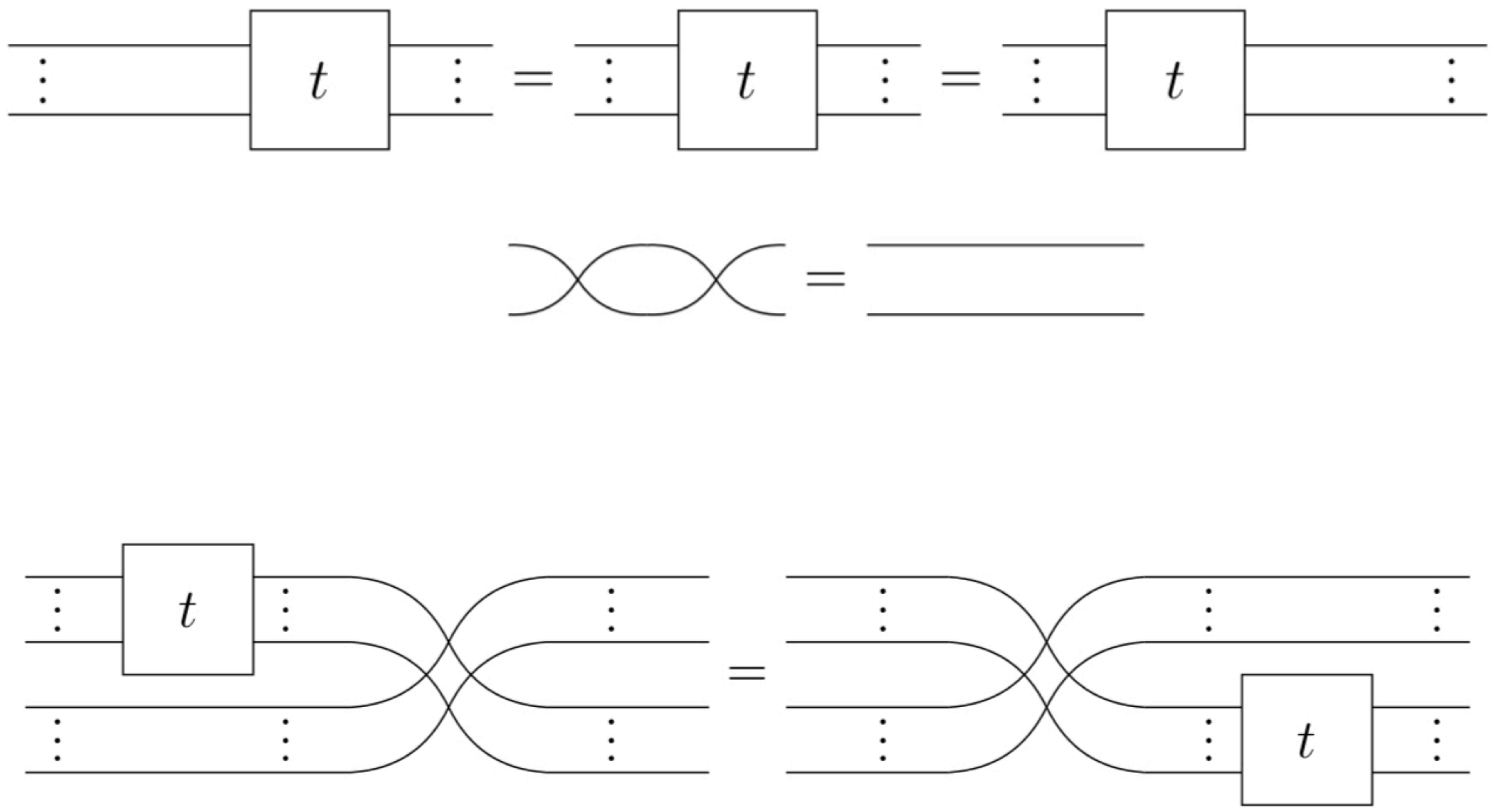


$l \xrightarrow{s} o$



$$n + l \xrightarrow{t \otimes s} m + o$$

PROPs



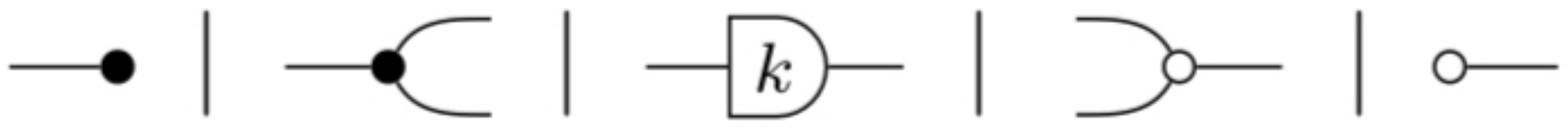
Our approach to build a theory

1. Build a category (a PROP) for the **syntax**
2. Choose an appropriate category for the **semantics**
3. Identify a set of sound and complete **axioms** for the theory

$$[\![c]\!] = [\![d]\!] \text{ if and only if } c = d$$

Hopf Algebras (HA)
The theory of k-Matrices

Syntax

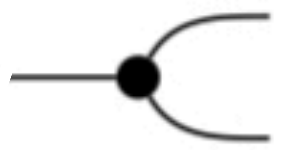


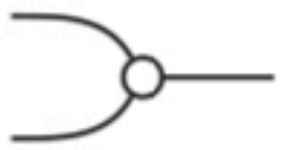
Semantics

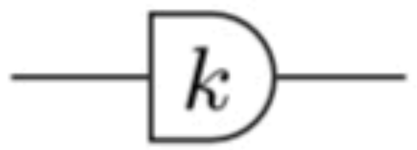
Mat_k

Hopf Algebras (HA)

The theory of k-Matrices

 $\longmapsto \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

 $\longmapsto \begin{pmatrix} 1 & 1 \end{pmatrix}$



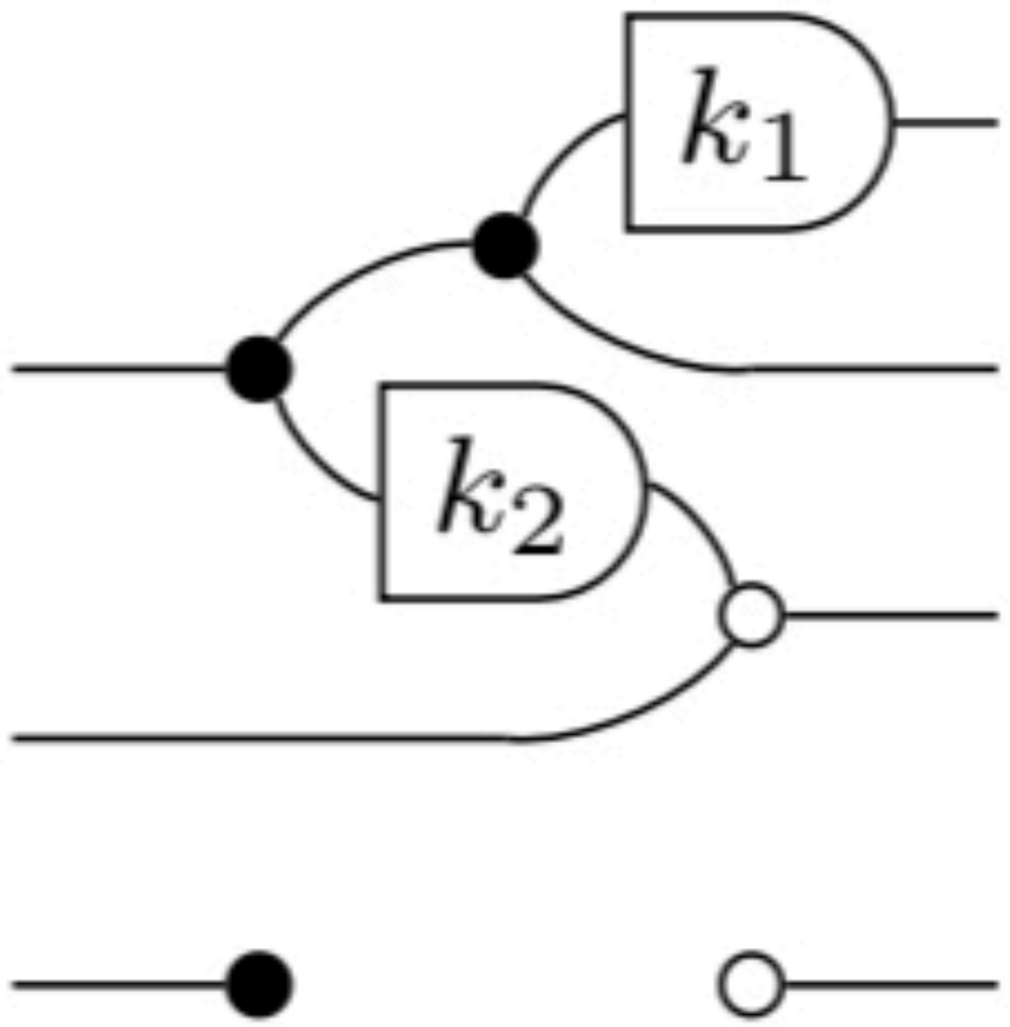
$\longmapsto \begin{pmatrix} k \end{pmatrix}$

 $\longmapsto\ !$

 $\longmapsto\ ¡$

Hopf Algebras (HA)

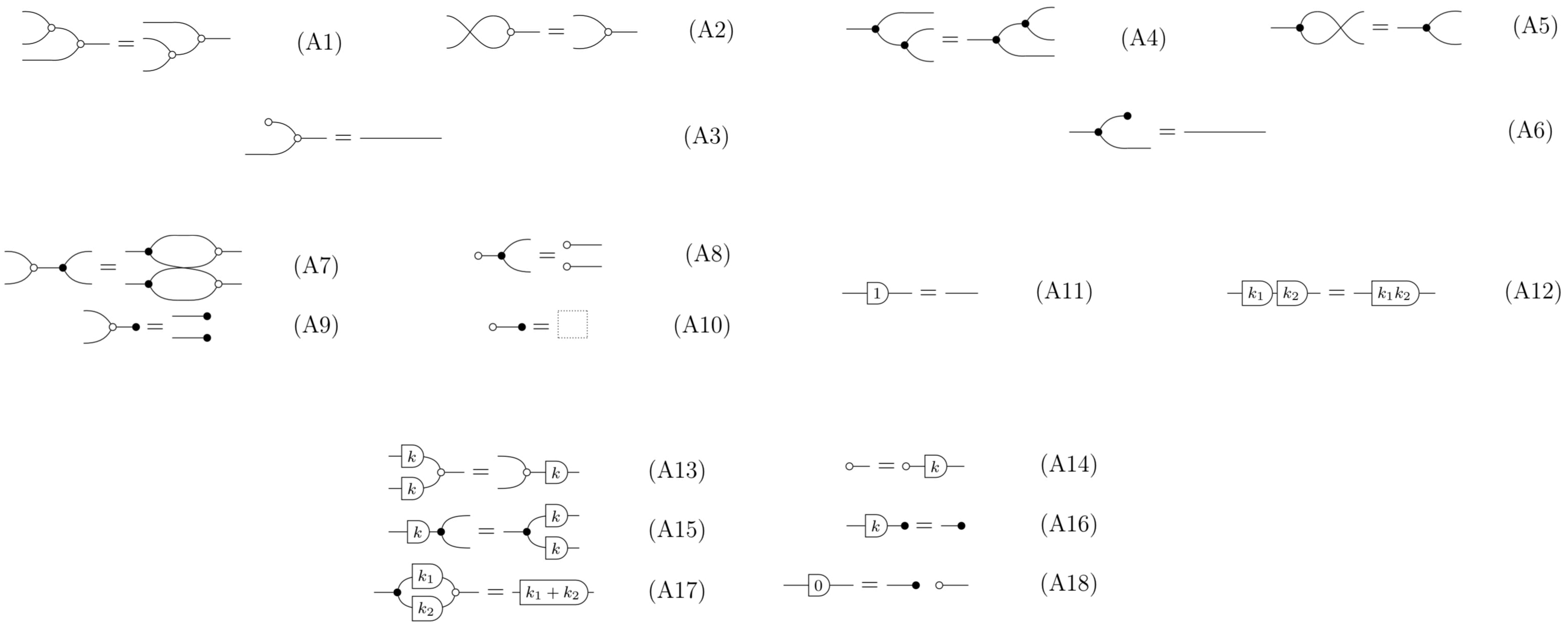
The theory of k-Matrices



$$\mapsto \begin{pmatrix} k_1 & 0 & 0 \\ 1 & 0 & 0 \\ k_2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hopf Algebras (*HA*)

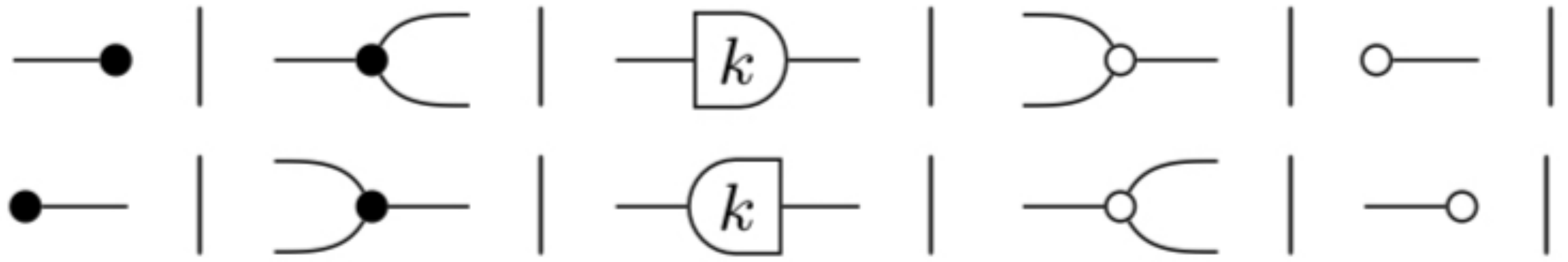
The theory of k-Matrices



Interacting Hopf Algebras (*IH*)

The theory of k-Linear relations

Syntax

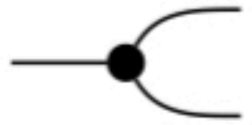


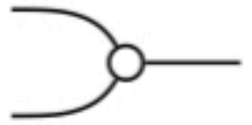
Semantics

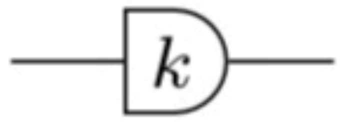
Rel_k

Interacting Hopf Algebras ($\mathbb{IH}$)

The theory of k-Linear relations

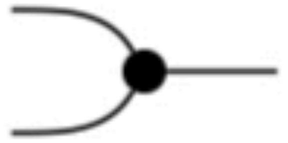
 $\longmapsto \{(x, \begin{pmatrix} x \\ x \end{pmatrix}) \mid x \in \mathsf{k}\}$

 $\longmapsto \{(\begin{pmatrix} x \\ y \end{pmatrix}, z) \mid x, y, z \in \mathsf{k}, z = x+y\}$

 $\longmapsto \{(x, kx) \mid x \in \mathsf{k}\}$

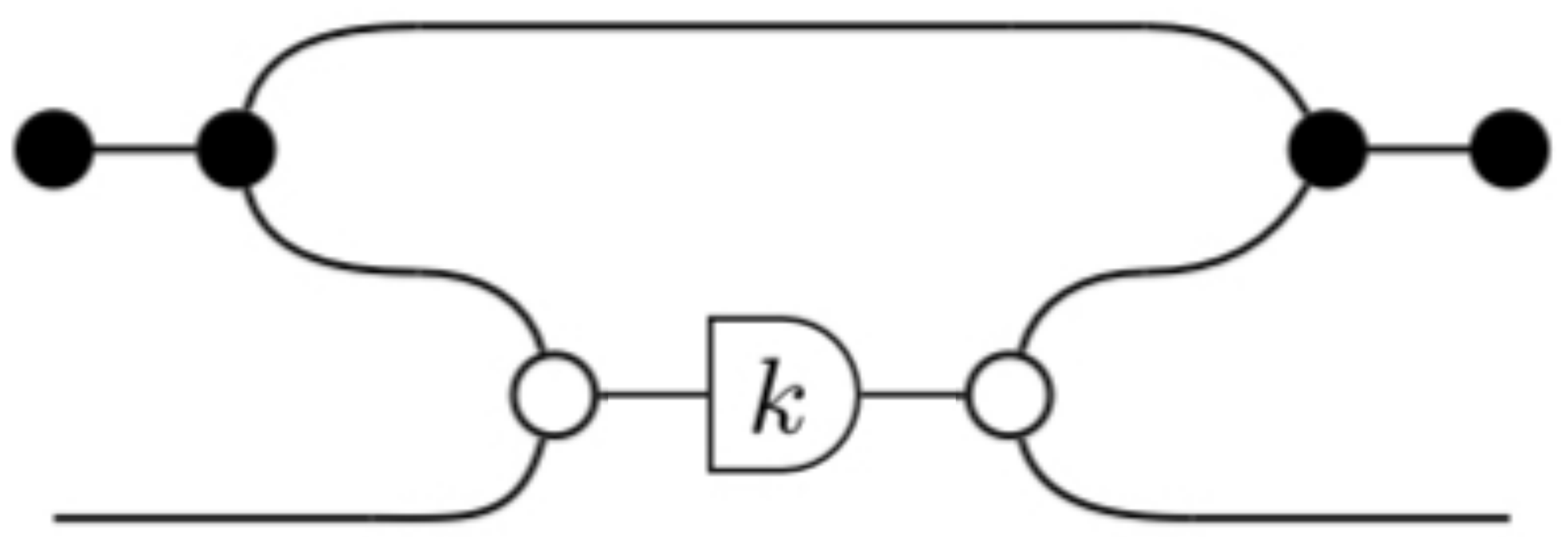
 $\longmapsto \{(x, \bullet) \mid x \in \mathsf{k}\}$

 $\longmapsto \{(\bullet, 0)\}$

 $\longmapsto \{(\begin{pmatrix} x \\ x \end{pmatrix}, x) \mid x \in \mathsf{k}\}$

Interacting Hopf Algebras (IH)

The theory of k-Linear relations

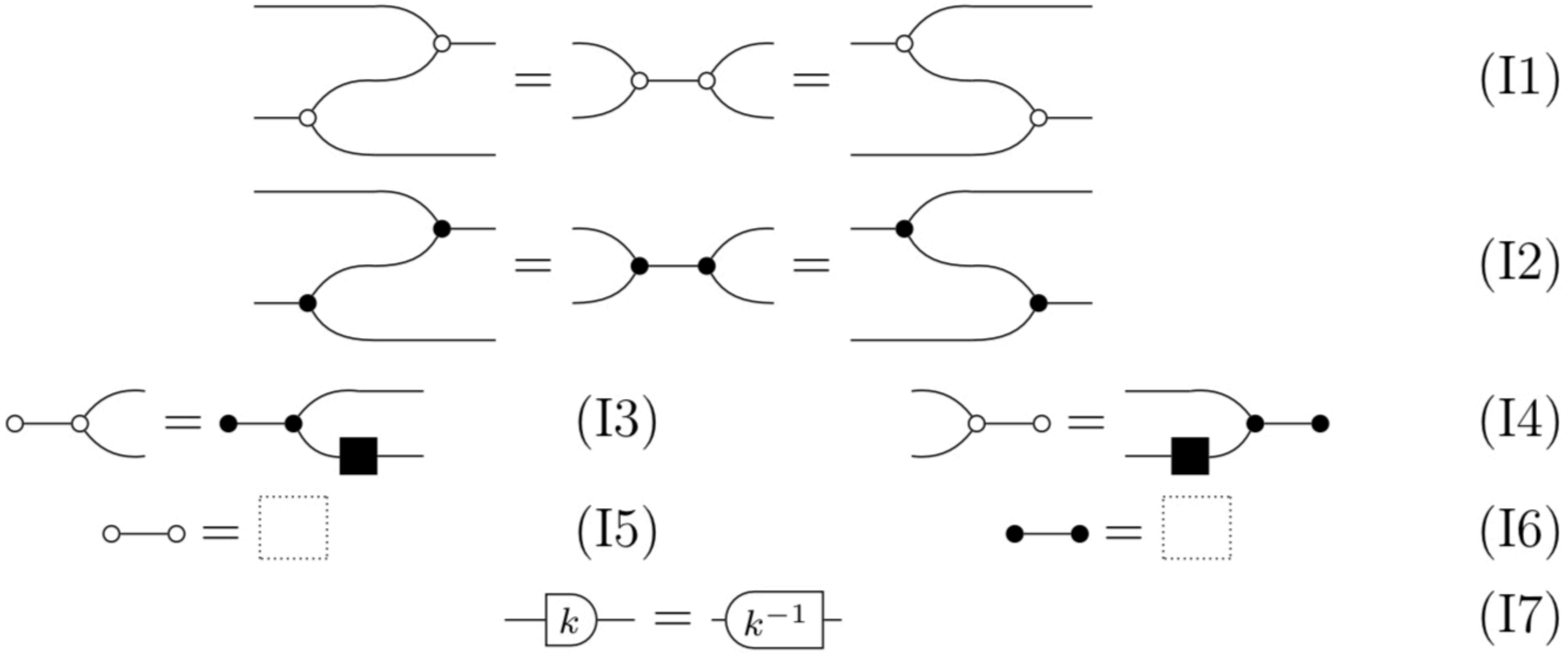


$$\longmapsto \quad \{(x, y) \mid \exists a \,.\, k(x + a) = y + a\}$$

Interacting Hopf Algebras ($\mathbb{IH}$)

The theory of k-Linear relations

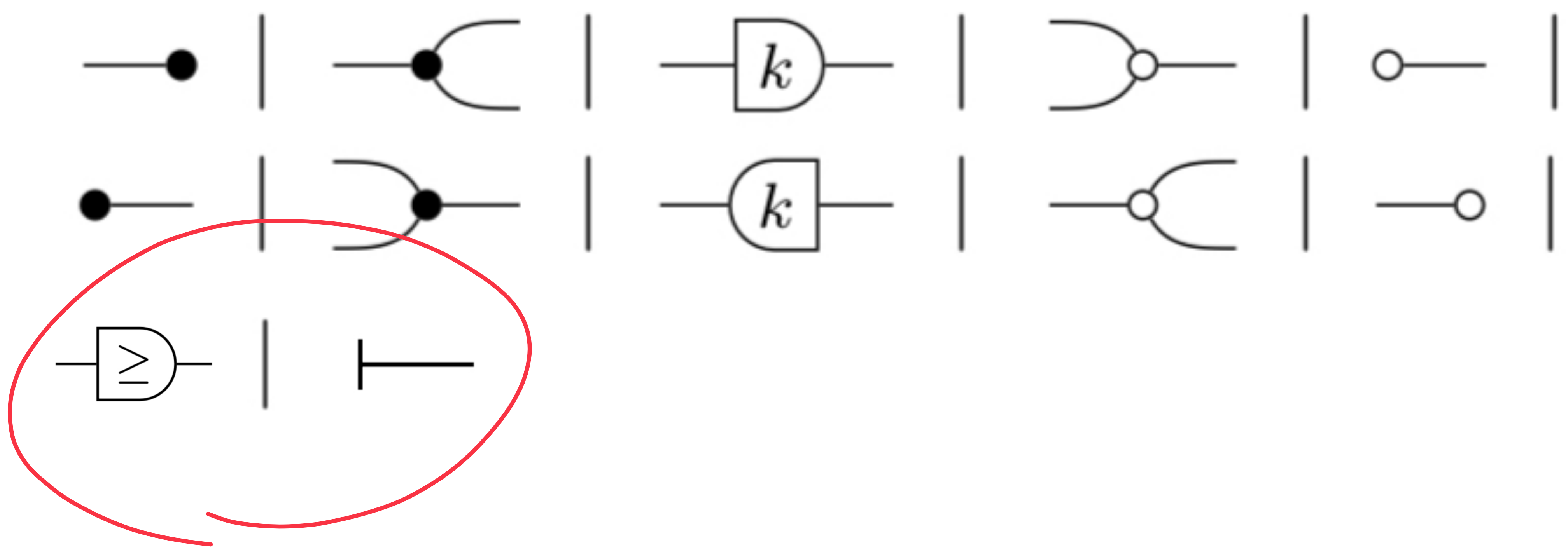
Axioms of $\mathbb{HA}$ +



Extending Interacting Hopf Algebras ($\mathcal{AIH}^+$)

The theory of k-Polyhedra

Syntax

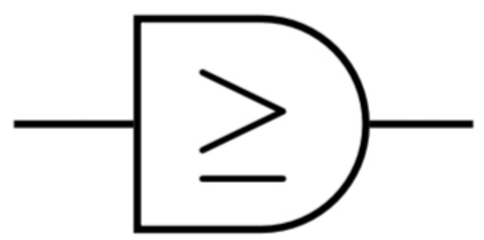


Semantics

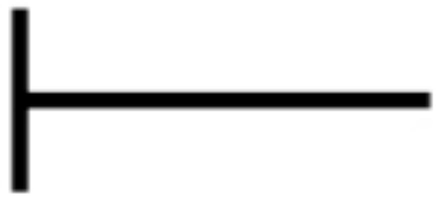
Rel_k

Extending Interacting Hopf Algebras (AIH^+)

The theory of k-Polyhedra



$$\longmapsto \{(x, y) \mid x \geq y\}$$



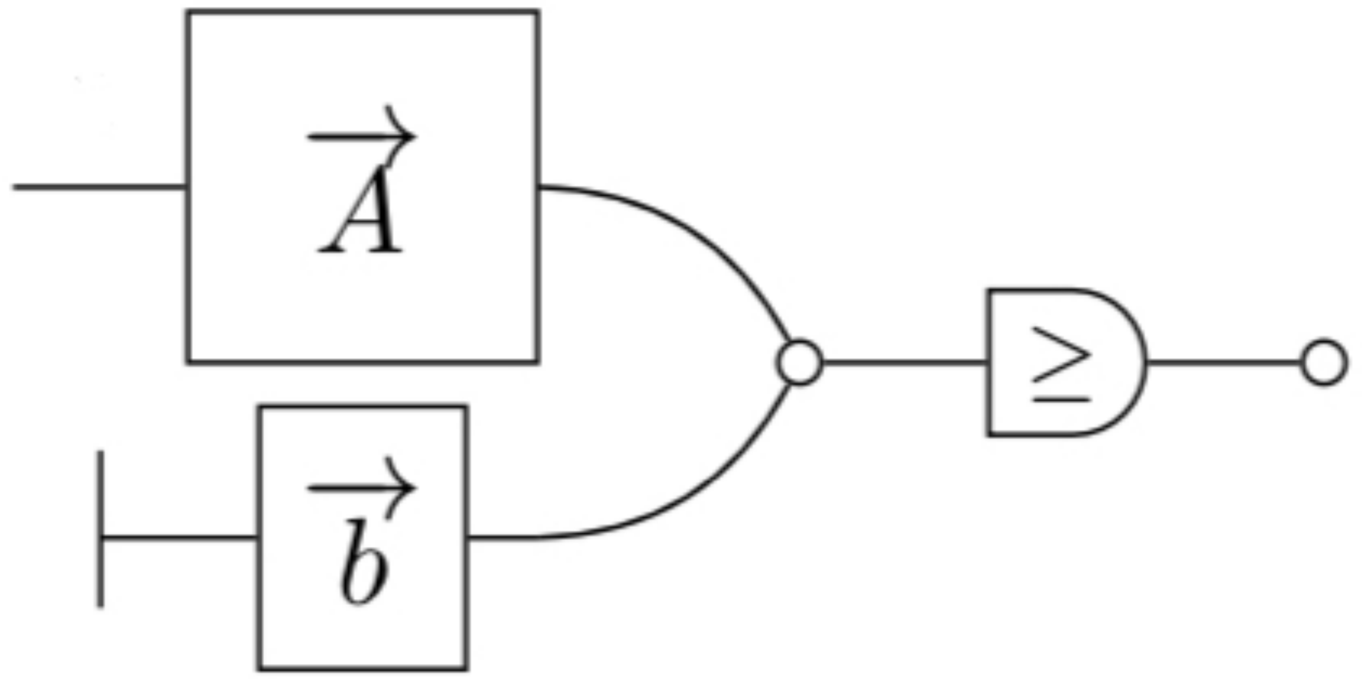
$$\longmapsto \{(\cdot, 1)\}$$

Polyhedra

$$P = \{x \in \mathsf{k}^n \,|\, Ax \geq b\}$$

P is the set of solutions of the system $Ax \geq b$

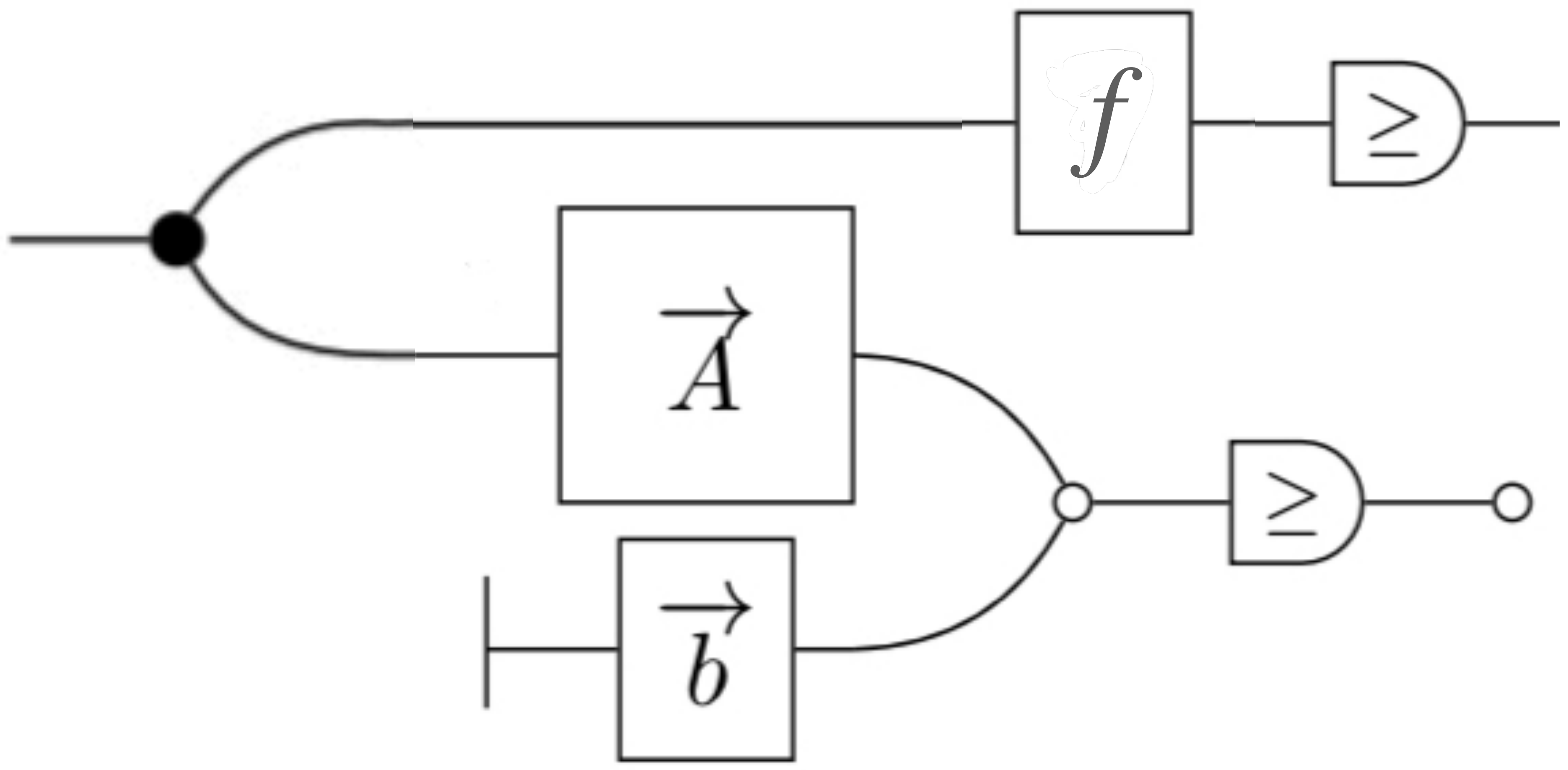
Polyhedron normal form (AIH^+)



$$\mapsto \{x \mid Ax + b \geq 0\}$$

Encoding Linear Programs

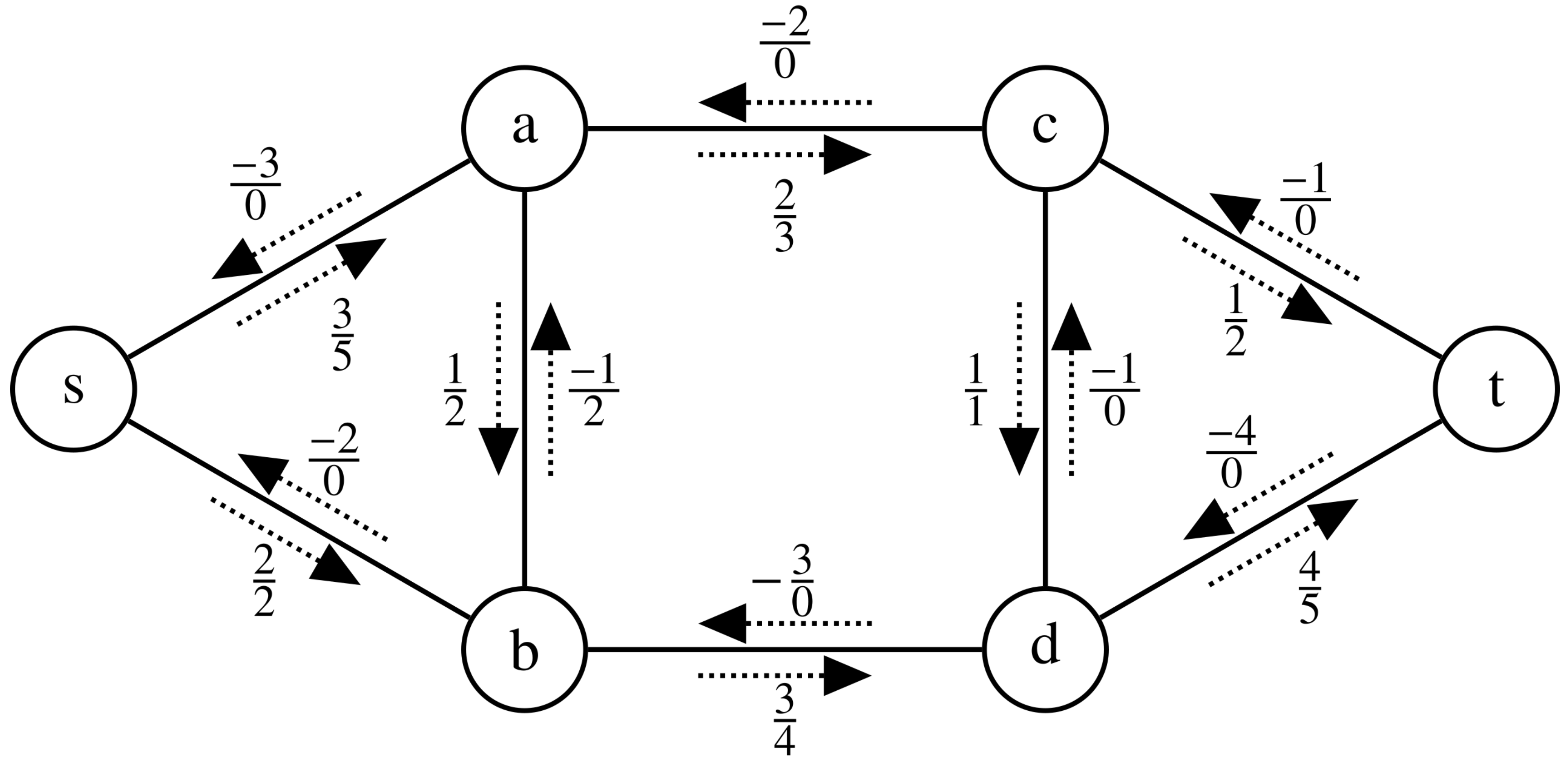
$$\min \; f(x)$$
$$Ax + b \geq 0$$



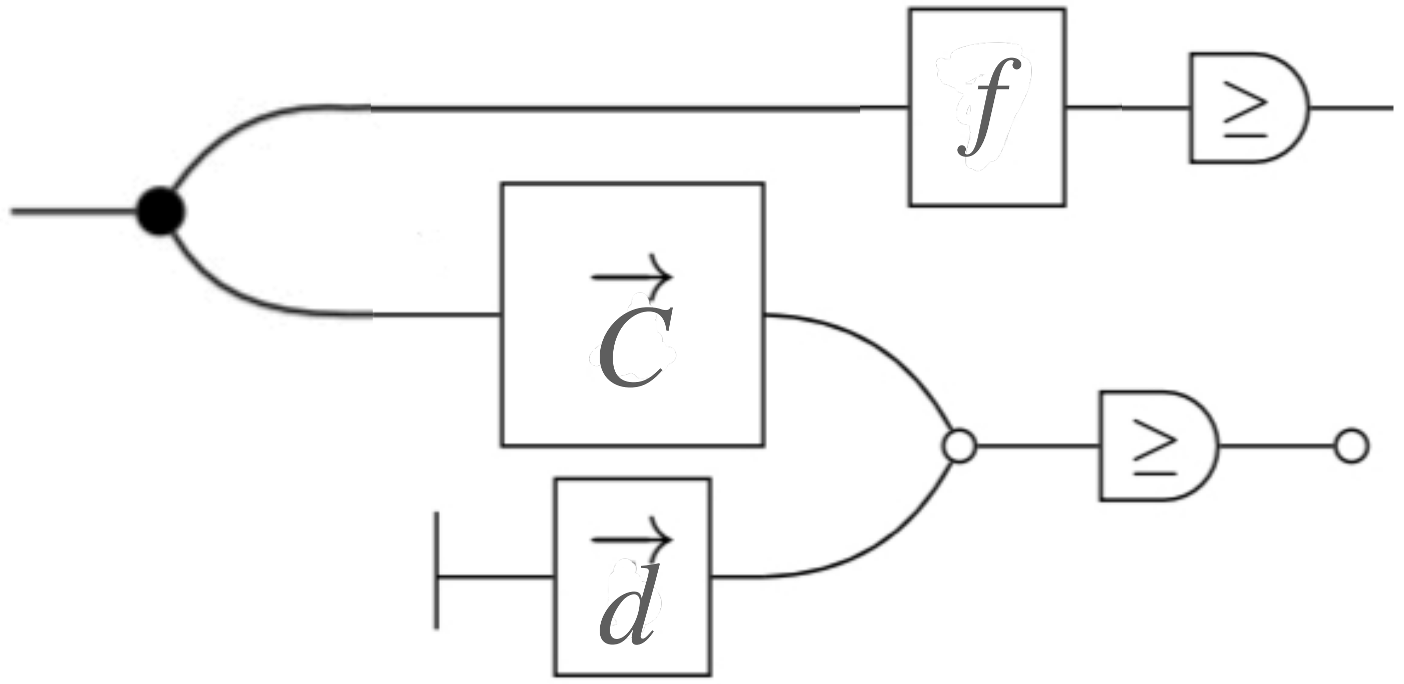
What we can do with AIH^+

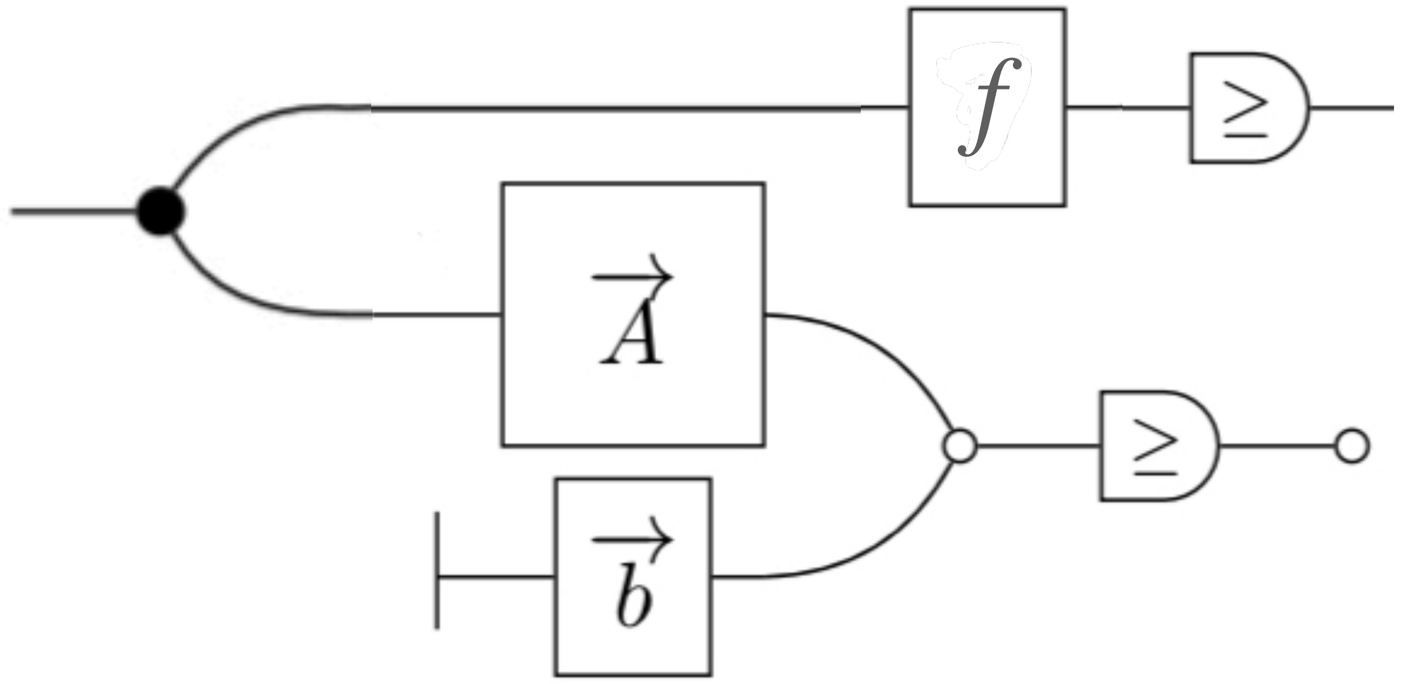
1. Encode any linear program into a diagram
2. Rewrite/simplify the program (or parts of it) according to the axioms of the theory
3. Identify equal linear programs

Flow networks



$$\min \; f(x)$$
$$Ax + b \geq 0$$





Conclusions

We gave an **algebraic/axiomatic** perspective on **linear programming**

References

Bonchi, Filippo, Paweł Sobociński, and Fabio Zanasi. "Interacting hopf algebras." *Journal of Pure and Applied Algebra* 221.1 (2017): 144-184.

Bonchi, Filippo, et al. "Graphical affine algebra." *2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. IEEE, 2019

Bonchi, F., Holland, J., Piedeleu, R., Sobociński, P., & Zanasi, F. (2019). Diagrammatic algebra: from linear to concurrent systems. *Proceedings of the ACM on Programming Languages*, *3*(POPL), 1-28.

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